Exercise 1
Let $X$ be a tangent vector field on a smooth manifold $M$, that is, $X$ is a map $M \to TM$ with $\pi \circ X = \text{Id}_M$, where $\pi : TM \to M$ is the projection map. Recall that, given a chart $\varphi : U \to V$ of $M$, the associated coordinate vector fields $\{ \frac{\partial}{\partial \varphi^1}, \ldots, \frac{\partial}{\partial \varphi^n} \}$ form a basis of $TM$ in each point of $U$, in particular any tangent vector field $X$ on $M$ can be written in the form $X = X^i \frac{\partial}{\partial \varphi^i}$ on $U$, where $X^1, \ldots, X^n : U \to \mathbb{R}$ are functions.

Show that $X$ is smooth as a map between manifolds if and only if the functions $X^1, \ldots, X^n : U \to \mathbb{R}$ are smooth for any chart.

Exercise 2
Let $f : M \to \mathbb{R}$ be a $C^1$ map on a compact smooth $n$-dimensional manifold, where $n \geq 1$.

1. Let $p \in M$ be a point. Show that, if $f$ reaches a maximum or a minimum at $p$, then $d_pf = 0$.

2. Show that the differential map of $f$ vanishes in at least two points in $M$.

3. Show that $f$ has exactly one critical value if and only if $f$ is constant.

Exercise 3
Let $M$ be a compact smooth $n$-dimensional manifold. By definition, a one-parameter group of diffeomorphisms on $M$ is a smooth map $\varphi : M \times \mathbb{R} \to M$, $(x, t) \mapsto \varphi_t(x)$, with $\varphi_0 = \text{Id}_M$ and $\varphi_t \circ \varphi_s = \varphi_{t+s}$ for all $s, t \in \mathbb{R}$.

1. Show that, given any one-parameter group of diffeomorphisms $(\varphi_t)_t$ on $M$, the map $X(x) := \left. \frac{d}{dt} \right|_{t=0}(\varphi_t(x))$ defines a smooth tangent vector field on $M$.

2. Conversely, show that, given any smooth vector field $X$ on $M$, there exists a unique one-parameter group of diffeomorphisms $(\varphi_t)_t$ on $M$ such that $\left. \frac{d}{dt} \right|_{t=0}(\varphi_t(x)) = X(x)$ for all $x \in M$.

Hint: First construct $\varphi_t(x)$ for fixed $x$ and $t$ close to 0 using the theorem of Picard-Lindelöf, then show that $t \mapsto \varphi_t(x)$ can be extended on $\mathbb{R}$.