Exercise 1
Let \((M, g)\) be a smooth compact Riemannian manifold. Show that every geodesic of \((M, g)\) is defined in \(\mathbb{R}\).

Exercise 2

1. Let \(M_1\) and \(M_2\) be two smooth surfaces in \(\mathbb{R}^3\), and assume that we have a smooth curve \(c : I \to \mathbb{R}^3\) with \(c(I) \subset M_1 \cap M_2\). Further we assume that \(T_{c(t)}M_1 = T_{c(t)}M_2\) for all \(t \in I\). Show that the parallel transports along \(c\) in \(M_1\) and in \(M_2\) coincide.

2. Given \(\theta \in ]0, 2\pi[\) let \(C := \{(r \cos \varphi, r \sin \varphi) : r \in ]0, \infty[, \varphi \in ]0, \theta[\} \subset \mathbb{R}^2\). Determine the parallel transport along the curve \(c_r : ]0, \theta[ \to C, t \mapsto (r \cos t, r \sin t)\), where \(r > 0\).

3. Deduce an explicit formula for the parallel transport along a circle of latitude \(t \mapsto (\cos t \cos \varphi, \sin t \cos \varphi, \sin \varphi)\) in \(S^2\), where \(\varphi \in ]-\pi/2, \pi/2[\).

Exercise 3 (Geodesics in hyperbolic space)

1. Let \(\phi : M \to M\) be an isometry of \((M, g)\). For any \(X \in TM\) let \(\gamma_X : I \to M\) be a geodesic with \(\frac{d}{dt}\gamma_X(0) = X\). Show: \(\phi(\gamma_X(t)) = \gamma_X(t)\) for all \(t \in I\) if and only if \(d\phi(X) = X\). If \(\gamma : I \to M\) is an arbitrary curve in \(M\) with \(\phi(\gamma(t)) = \gamma(t)\) for all \(t \in I\), then \(d\phi(\frac{d}{dt}\gamma(t)) = \frac{d}{dt}\phi(\gamma(t))\) for all \(t \in I\).

2. Let \(\mathbb{H}^n := \{x \in \mathbb{R}^{n+1} : \langle x, x \rangle = -1\) and \(x_0 > 0\}\) denote the \(n\)-dimensional hyperbolic space (see Exercise no. 4 in Sheet 5). We identify \(T_x\mathbb{H}^n\) with \(x^\perp := \{V \in \mathbb{R}^{n+1} : \langle V, x \rangle = 0\}\). For \(V \in x^\perp\) we define \(\|V\| := \sqrt{\langle V, V \rangle}\). For \(V \in x^\perp \setminus \{0\}\) show that
\[
\gamma_{x,V}(t) := \cosh(\|V\| t)x + \sinh(\|V\| t) \frac{V}{\|V\|}
\]
is a curve in \(\mathbb{H}^n\).

3. Determine a linear map \(\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}\) with \(\langle \Phi(\cdot, \cdot) \rangle = \langle \cdot, \cdot \rangle\) such that its fixed point set is the plane spanned by \(x \in \mathbb{H}^n\) and \(V \in x^\perp \setminus \{0\}\). Show that its restriction to \(\mathbb{H}^n\) defines an isometry \(\phi : \mathbb{H}^n \to \mathbb{H}^n\). What is the fixed point set?

4. Conclude that \(\frac{d}{dt}\gamma_{x,V}(t) = f(t)\gamma_{x,V}(t)\) for all \(t \in \mathbb{R}\) and for a suitable function \(f\).
5. Show that $\gamma_{x,V}$ is a geodesic. *Hint: Calculate $||\dot{\gamma}_{x,V}(t)||$. Are all non-constant geodesics in $\mathbb{H}^n$ of this form?*

**Exercise 4**

Does there exist a Riemannian metric

1. on $\mathbb{R}^2$ such that all circles can be parametrized as geodesics?
2. on $\mathbb{R}^2 \setminus \{0\}$ such that all circles centered at 0 can be parametrized as geodesics?
3. on $\mathbb{R}^2 \setminus \{0\}$ such that all circles centered at 0 can be parametrized as geodesics but *no* ray through 0 is a geodesic?

Justify each of your answers.