Exercise 1
Let $\gamma : [a, b] \to M$ be a piecewise $C^1$ curve on a smooth Riemannian manifold $(M, g)$.

(a) Prove that $L[\gamma]^2 \leq 2(b-a) \cdot E[\gamma]$, where $E[\gamma] := \frac{1}{2} \int_a^b g(\dot{\gamma}, \dot{\gamma}) dt$ is the energy of the curve $\gamma$.

(b) Show that $L[\gamma]^2 = 2(b-a) \cdot E[\gamma]$ holds if $\gamma$ is parametrized proportionally to arc-length.

Exercise 2
Let $M$ be a smooth submanifold of $\mathbb{R}^k$.

(a) Show that, if $M$ is closed, then $M$ is complete.

(b) Show that the converse statement is wrong.

Exercise 3
Let $(M, g)$ be a connected complete non-compact Riemannian manifold and $p \in M$ be a point.

(a) Show that there exists a sequence $(p_i)_{i \in \mathbb{N}}$ in $M$ such that $d(q, p_i) \to \infty$.

(b) Show that, for each $i \in \mathbb{N}$, there exist $X_i \in T_p M$ and $r_i \in [0, \infty]$ with $g_p(X_i, X_i) = 1$ and $p_i = \exp_p(r_i X_i)$.

(c) Show that the sequence $(X_i)_{i \in \mathbb{N}}$ admits a converging subsequence and deduce that there exists a ray $\gamma : [0, \infty) \to M$ in $(M, g)$ with $\gamma(0) = p$.

Exercise 4
Let $M$ be a connected $m$-dimensional manifold, and assume that $N \subset M$ is an $n$-dimensional submanifold, i.e. for every $p \in N$ there is a chart $\phi : U \to V \subset \mathbb{R}^m$, $p \in U$ such that $\phi(U \cap N) = V \cap (\mathbb{R}^n \times \{0\})$. Let $g$ be a Riemannian metric on $M$, such that $(M, g)$ is complete, and assume that $N$ is a closed (as a subset of $M$). Fix a point $q \in M$.

(a) Show the existence of a point $p \in N$ with $d(q, p) = d(q, N)$, where $d(q, N) := \inf_{x \in N} \{d(q, x)\}$. Is $p$ unique? Justify your answer.

(b) Prove that there is a geodesic $\gamma$ from $q$ to $p$ with length $L[\gamma] = d(q, p)$.

(c) Show that $\gamma$ meets $N$ orthogonally.