Exercise 1 Let \((M^n, g)\) be a Riemannian manifold and \((U_\varphi, \varphi)\) be a chart on \(M\).

(a) Show that the components \(R^l_{ijk} := d\varphi^l \left( R(\frac{\partial}{\partial \varphi^i}, \frac{\partial}{\partial \varphi^j}) \frac{\partial}{\partial \varphi^k} \right)\) of the curvature tensor \(R\) of \(g\) on \(U_\varphi\) are given in terms of the Christoffel symbols associated to \((U_\varphi, \varphi)\) by

\[
R^l_{ijk} = \frac{\partial \Gamma^l_{kj}}{\partial \varphi^i} - \frac{\partial \Gamma^l_{ki}}{\partial \varphi^j} + \sum_{m=1}^{n} \left( \Gamma^l_{mi} \Gamma^m_{kj} - \Gamma^l_{mj} \Gamma^m_{ki} \right).
\]

(b) Deduce that, if \(\text{ric} = \sum_{i,j=1}^{n} \text{ric}_{ij} d\varphi^i \otimes d\varphi^j\) denotes the decomposition of the Ricci-tensor of \(g\), then

\[
\text{ric}_{ij} = \sum_{k=1}^{n} R^k_{ij} = \sum_{k=1}^{n} \left( \frac{\partial \Gamma^k_{ji}}{\partial \varphi^k} - \frac{\partial \Gamma^k_{kj}}{\partial \varphi^i} + \sum_{m=1}^{n} \left( \Gamma^k_{mk} \Gamma^m_{ji} - \Gamma^k_{mi} \Gamma^m_{jk} \right) \right).
\]

Exercise 2
Let \((M := M_1 \times M_2, g := g_1 \oplus g_2)\) be the product of two Riemannian manifolds as in Exercise 3(b) of Sheet no. 10.

(a) Show that the Levi-Civita connection \(\nabla\) of \((M, g)\) is given by

\[
\nabla_{(X_1, X_2)}(Y_1, Y_2) = \nabla^{M_1}_{X_1} Y_1 + \nabla^{M_2}_{X_2} Y_2 + \partial_{X_1} Y_2 + \partial_{X_2} Y_1,
\]

for all \(X_1, Y_1 \in \Gamma(\pi_1^* TM_1), X_2, Y_2 \in \Gamma(\pi_2^* TM_2)\) and where \(\partial_{X_i} Y_2\) (resp. \(\partial_{X_2} Y_1\)) denotes the usual derivative (make sense of this).

(b) Deduce that the curvature tensor \(R\) of \((M, g)\) is given by

\[
R^M_{X,Y} Z = R^{M_1}_{X_1,Y_1} Z_1 + R^{M_2}_{X_2,Y_2} Z_2,
\]

for all \(X_i, Y_i, Z_i \in T_x M_i, i = 1, 2\), where \(X := X_1 + X_2, Y := Y_1 + Y_2\) and \(Z := Z_1 + Z_2\). (Here we write \(R_{X,Y}\) instead of \(R(X, Y)\) as this is better for typesetting in this context.)

(c) Calculate the Ricci tensor and the scalar curvature of \(M\) in terms of the Ricci tensor and the scalar curvature of \(M_1\) and \(M_2\).

Exercise 3
Let \(E, F \to M\) be (real or complex) vector bundles with connections \(\nabla^E, \nabla^F\) over a given manifold \(M\) and \(x \in M\) be a point. Prove the following identities:
(a) The curvature tensor of the connection $\nabla^E \oplus \nabla^F$ on $E \oplus F \to M$ is given by
\[ R^{E\oplus F}_{X,Y} = R^E_{X,Y} \oplus R^F_{X,Y}, \]
for all $X, Y \in T_x M$.

(b) The curvature tensor of the tensor connection on $E \otimes F \to M$ as defined in Exercise 3 of Sheet no. 11 is given by
\[ R^{E\otimes F}_{X,Y} = R^E_{X,Y} \otimes \text{Id}_F + \text{Id}_E \otimes R^F_{X,Y}, \]
for all $X, Y \in T_x M$.

(c) The curvature tensor of the dual bundle $E^* \to M$ endowed with the induced connection is given by
\[ (R^{E^*}_{X,Y}\alpha)(V) = -\alpha(R^E_{X,Y}V), \]
for all $X, Y \in T_x M$, for all $V \in E_x$ and $\alpha \in E^*_x$.

**Exercise 4**

Let $(M^n, g)$ be a Riemannian manifold and denote by $\nabla$ resp. $R$ the Levi-Civita connection resp. the Riemannian curvature tensor of $(M^n, g)$. Let the $(0,4)$-tensor $\tilde{R}$ be defined by $\tilde{R}(X,Y,Z,W) := g(R(X,Y)Z,W)$, for all $X,Y,Z,W \in T_x M$ and $x \in M$.

(a) Let $x \in M$ be a point. Prove that the following identities are satisfied:
for all $X, Y, Z, T, U \in T_p M$,
\[ (\nabla_X \tilde{R})(Y,Z,T,U) = -(\nabla_X \tilde{R})(Z,Y,T,U) = -(\nabla_X \tilde{R})(Y,Z,U,T) = (\nabla_X \tilde{R})(T,U,Y,Z). \]

(b) For a given tensor field $A \in \Gamma(T^* M \otimes T^* M)$ let the *divergence* of $A$ be defined by
\[ \text{div}(A)(X) := \sum_{j=1}^n (\nabla_{E_j} A)(E_j, X) \quad \forall X \in TM, \]
where $\{E_j\}_{1 \leq j \leq n}$ is a local orthonormal basis of $TM$, that is, $g(E_i, E_j) = \delta^i_j$. Prove using the second Bianchi identity:
\[ \text{div}(\text{ric}) = \frac{1}{2} \text{scal}. \]

(Hint: for a given point $x \in M$, the basis $\{E_j\}_{1 \leq j \leq n}$ can be chosen such that $(\nabla E_i)|_x = 0$ holds; how can this be done?)

(c) *Application*: Assuming $n \geq 3$, the manifold $M$ connected and the existence of a smooth function $f : M \to \mathbb{R}$ with $\text{ric} = f \cdot g$ on $M$, prove that $f$ is constant on $M$. Such a Riemannian manifold is then called *Einstein*.